Mimetic Staggered Discretization of Incompressible Navier–Stokes for Barycentric Dual Mesh

René Beltman, Martijn J. H. Anthonissen and Barry Koren

Abstract A staggered discretization of the incompressible Navier–Stokes equations is presented for polyhedral non orthogonal nonsmooth meshes admitting a barycentric dual mesh. The discretization is constructed by using concepts of discrete exterior calculus. The method strictly conserves mass, momentum and energy in the absence of viscosity.

Keywords Mimetic finite-volume discretizations · Barycentric dual mesh

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1 Introduction

In staggered methods the incompressible Navier–Stokes equations are discretized in terms of the normal velocity components at the cell faces and pressure variables in the cell-centers. Staggered mesh methods were introduced for Cartesian meshes by Harlow and Welch in the form of the MAC scheme [6]. The staggering allows for an efficient discretization of the divergence-free condition, leading to exact conservation of mass. It was subsequently shown [7] that, besides momentum and mass, the staggered Cartesian discretization also conserves the secondary quantities vorticity and kinetic energy, in the inviscid case.

The staggered mesh method was subsequently extended to unstructured meshes [5, 9]. In this formulation the orthogonality properties of a Delaunay-Voronoi dual

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mesh were exploited. Perot [10] showed that, on unstructured meshes, both for a discretization of the momentum equation in divergence-form and a discretization in rotation-form, kinetic energy is conserved. However, for the divergence-form conservation of momentum was proved but not conservation of vorticity, and, for the rotation-form conservation of vorticity was shown to be satisfied, but not conservation of momentum. Recently, a variational formulation of the MAC scheme was generalized to nonconforming meshes and proved to converge [3].

In most of the aforementioned cases, the primal mesh admits a circumcentric dual mesh. The circumcentric dual mesh has desirable orthogonality properties that allow for simple interpolation between the primal and dual meshes. For many meshes a circumcentric dual mesh does not exist. Such a situation is encountered in, for example, cut-cell methods [4]. In such cases a barycentric dual mesh can be used, although this type of dual mesh is harder to deal with, because it lacks orthogonality properties.

In this work we will present a barycentric discretization. We will discretize the divergence-form of the Navier–Stokes equations, given by

$$\frac{\partial \underline{u}}{\partial t} = (\underline{u} \cdot \nabla)\underline{u} - \frac{1}{\rho}\nabla p + \nu \Delta \underline{u}, \tag{1a}$$

$$0 = \nabla \cdot \underline{u}. \tag{1b}$$

The discretization presented in [12] will be generalized to polyhedral meshes. This will be done by showing that the mimetic inner product matrices [2] can be interpreted as discrete Hodge operators and by using them as such. They allow for polyhedral volumes with a varying number of faces. Furthermore, we will complete the dual mesh to a cell-complex and show that a discretization on this cell-complex leads to a method that conserves mass, momentum and energy (in the absence of viscosity) in the interior of the domain and also reproduces accurate boundary fluxes of these quantities. The discretization has a narrow stencil for the orthogonal part of the mesh.

2 Primal-Dual Mesh Structure and Discrete Exterior Calculus

In the continuous setting, conservation statements, like conservation of energy, can be derived from the primary equations by using fundamental properties of the continuous differential operators. Examples of this are that the curl of a gradient is always zero and the divergence of a curl is always zero. If the fundamental properties of the continuous differential operators can be transferred to the discrete setting, then it is possible to derive discrete conservation properties by similar arguments as in the continuous case. We will discretize in such a way that properties of the continuous differential operators are transferred to the discrete setting as much as possible.

2.1 The Primal Mesh and Incidence Matrices

The mesh $\mathscr{G} := \{\mathscr{P}, \mathscr{L}, \mathscr{S}, \mathscr{V}\}$ consists of a set of points $\mathscr{P}$, lines $\mathscr{L}$, surfaces $\mathscr{S}$ and volumes $\mathscr{V}$. The volumes $\mathscr{V}$ are polyhedra that exactly fill the flow domain Ω. The intersection of two volumes in $\mathscr{V}$ is either empty or it is a polygon part of the boundary of both. The set $\mathscr{S}$ is the union of the polygons making up the boundary of all the volumes in $\mathscr{V}$. Similarly, $\mathscr{L}$ is the union of the different line segments making up the boundaries of the polygons in $\mathscr{S}$ and $\mathscr{P}$ is the union of all endpoints of lines in $\mathscr{L}$.

We discretize the velocity field by integrating over the polygonal faces $s \in \mathscr{S}$:

$$u_s^{(2)} := \int_s \underline{u} \cdot \mathrm{d}\underline{A},$$

where $\mathrm{d}\underline{A}$ is an infinitesimal oriented surface area and the superindex indicates the dimension of s. The numbers $u_s^{(2)}$, $s \in \mathscr{S}$, are the discrete variables and, if we number the elements in $\mathscr{S}$, we can order them in a vector $\mathbf{u}^{(2)}$. We denote the space of possible $\mathbf{u}^{(2)}$ by $C^{(2)} = \mathbb{R}^{N_s}$, where N_s is the number of elements in $\mathscr{S}$. In a similar vein we can discretize a vector field by integrating it along the lines in $\mathscr{L}$ and define a space $C^{(1)}$. By integrating and evaluating a scalar function on the elements of $\mathscr{V}$ and $\mathscr{P}$, respectively, we can discretely represent this function on the volumes or points of our mesh and analogously define the spaces $C^{(3)}$ and $C^{(0)}$.

The integrated discrete variables are also known as discrete forms [8] or cochains [1]. They allow to discretize the divergence, gradient and curl (or really the exterior derivative) in such a way that the generalized Stokes theorem is valid in the finite number of situations provided by the mesh. To be able to define the discretizations of the divergence, gradient and curl we need to give an orientation to the elements in $\mathscr{G}$. The choice of orientation is arbitrary. Examples of oriented mesh elements are shown in Fig. 1. Suppose we are given the surface fluxes $u_s^{(2)}$. Using the divergence theorem we can determine the divergence of $\underline{u}$ integrated over all $v \in \mathscr{V}$:

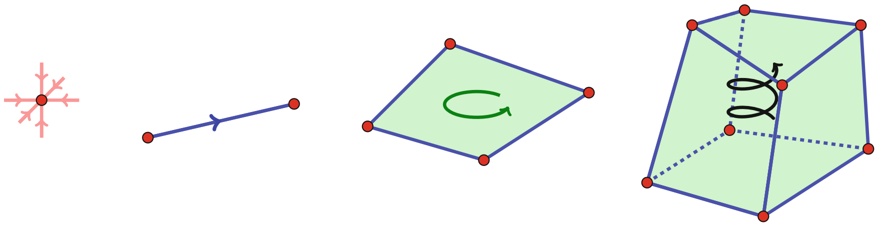

Fig. 1 A point $p \in \mathscr{P}$ is either classified as a sink (ingoing *arrows*) or a source (outgoing *arrows*), a *line* $l \in \mathscr{L}$ is oriented by a direction along the *line*, a face $s \in \mathscr{S}$ by a sense of rotation in its plane and a volume $v \in \mathscr{V}$ by a right- or left-hand-rule

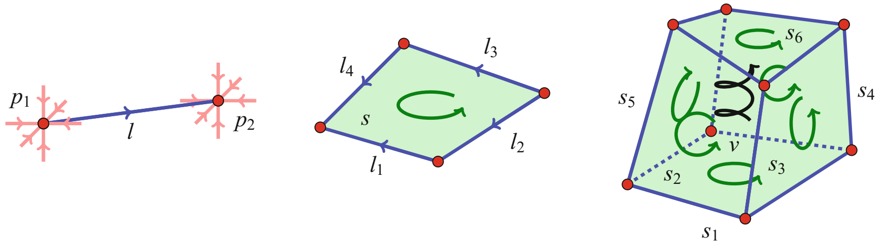

Fig. 2 For the line the orientations of p_2 and l agree, therefore $\alpha_l(p_2) = 1$. Similarly we have $\alpha_l(p_1) = -1$, $\alpha_s(l_1) = \alpha_s(l_2) = -1$, $\alpha_s(l_3) = \alpha_s(l_4) = 1$. Volume v has right-handed orientation indicated by a helix and we find $\alpha_v(s_1) = \alpha_v(s_4) = -1$ and $\alpha_v(s_2) = \alpha_v(s_3) = \alpha_v(s_5) = \alpha_v(s_6) = 1$

$$\int_v \nabla \cdot \underline{u} \, \mathrm{d}V = \sum_{s \in \partial v} \alpha_v(s) u_s^{(2)},$$

where $\alpha_v(s) = 1$ if the orientations of s and v agree and $\alpha_v(s) = -1$ otherwise. Examples are given in Fig. 2. In matrix notation we can express this for all $v \in \mathscr{V}$ at once as $\mathbf{d}^{(3)} = \mathbb{D}^{(3,2)}\mathbf{u}^{(2)}$, where the entry of $\mathbf{d}^{(3)}$ corresponding to $v \in \mathscr{V}$ is $d_v^{(3)} := \int_v \nabla \cdot \underline{v} \, \mathrm{d}V$ and

$$\mathbb{D}^{(3,2)}_{v,s} := \begin{cases} +1 & \text{if } s \in \partial v \text{ and the orientations of } s \text{ and } v \text{ agree,} \\ -1 & \text{if } s \in \partial v \text{ and the orientations of } s \text{ and } v \text{ disagree,} \\ 0 & \text{if } s \notin \partial v. \end{cases}$$

We see that the incidence matrix $\mathbb{D}^{(3,2)}$ gives the integral of the divergence of a vector field $\underline{u}$ over the volumes when it is applied to the discrete representation of this vector field on the surfaces. Similarly, using the fundamental theorem of calculus, the discrete representation of a function on $\mathscr{P}$ can be used to determine the integral of the gradient over the lines in $\mathscr{L}$. Suppose $\mathbf{q}^{(0)} \in C^{(0)}$ is this discrete representation, i.e., $q_p^{(0)} := q\big|_p$ then $[\mathbb{D}^{(1,0)}\mathbf{q}^{(0)}]_l = \int_l \nabla q \cdot \mathrm{d}\underline{l}$, where $\mathbb{D}^{(1,0)}_{l,p} = +1$, if $p \in \partial l$ and their orientations agree, $\mathbb{D}^{(1,0)}_{l,p} = -1$ if $p \in \partial l$ and their orientations disagree and $\mathbb{D}^{(1,0)}_{l,p} = 0$ if $p \notin \partial l$. A similarly defined matrix $\mathbb{D}^{(2,1)}$ returns the integral of the curl over surfaces in $\mathscr{S}$ of a vector field when applied to a discretization of this vector field on $\mathscr{L}$, representing an exact discretization of the Kelvin-Stokes theorem. The incidence matrices have the properties $\mathbb{D}^{(2,1)}\mathbb{D}^{(1,0)}\mathbf{a}^{(0)} = \mathbf{0}^{(2)}$ for all $\mathbf{a}^{(0)} \in C^{(0)}$ and $\mathbb{D}^{(3,2)}\mathbb{D}^{(2,1)}\mathbf{b}^{(1)} = \mathbf{0}^{(3)}$ for all $\mathbf{b}^{(1)} \in C^{(1)}$ representing the fact that the curl of a gradient and the divergence of a curl are zero [1].

2.2 The Dual Mesh and Discrete Hodge Operators

Primal mesh and incidence matrices alone are not sufficient to discretize the equations. We introduce a dual mesh which allows to conveniently interpolate between discrete variables defined on mesh elements of dimension k and dual mesh elements of dimension $3-k$. Using these interpolations we can, for example, apply the incidence matrix corresponding to the curl twice, once on the primal mesh and, after interpolation to the dual mesh, once on the dual mesh. This then allows for the construction of discretizations of higher order differential operators like the Laplacian. Moreover, the interpolation between the primal and dual mesh introduces the metric aspects of the differential equation in the discrete setting. The discrete operations on only the primal (or dual) mesh, as indicated by the ones and zeros of the incidence matrices, only depend on the topology of the mesh and not on the lengths, areas or volumes of the mesh elements. These metrical notions only play a role in the interpolation between the primal and dual mesh. This interpolation is also the place where the discretization error enters the method.

The dual mesh $\tilde{\mathscr{G}} := \{\tilde{\mathscr{V}}, \tilde{\mathscr{S}}, \tilde{\mathscr{L}}, \tilde{\mathscr{P}}\}$ consists of the set of points $\tilde{\mathscr{V}}$ which are dual to the volumes in $\mathscr{V}$, the set of lines $\tilde{\mathscr{S}}$ dual to the surfaces in $\mathscr{S}$, etc. Everything related to the dual mesh will be given a tilde. In the introduction we mention the circumcentric dual mesh, constructed by connecting the circumcenters of neighboring primal volumes, and the barycentric dual mesh, constructed by connecting the barycenters of every primal volume with the barycenters of the primal surfaces that constitute the boundary of that primal volume. It should be noted that the line elements $\tilde{\mathscr{S}}$ of the barycentric dual mesh consist of two straight line segments. The dual mesh elements are given an outer orientation, i.e. an orientation of their complement in the ambient Euclidean three-dimensional space. This orientation will be chosen to coincide with the orientation of their corresponding primal cells. For the dual mesh, analogously to the primal mesh, we define discrete spaces $C^{(\tilde{k})}$, $k = 0, 1, 2, 3$, and incidence matrices $\mathbb{D}^{\widetilde{(k+1,\tilde{k})}}$, $k = 0, 1, 2$. Note that $C^{(k)} = C^{\widetilde{(3-k)}}$, because of the bijection between $\tilde{\mathscr{G}}$ and $\mathscr{G}$. If the dual mesh elements are numbered in the same way as the primal mesh elements and the outer orientation for the dual mesh is chosen to coincide with the primal mesh, then $\mathbb{D}^{\widetilde{(k+1,\tilde{k})}} = \mathbb{D}^{(3-k,3-k-1),T}$, $k = 0, 1, 2$, see [13].

The primal mesh is a so-called cell-complex, because the boundary of every element in $\mathscr{G}$ is either a union of lower dimensional elements in $\mathscr{G}$ or empty. This property implies that a discrete divergence theorem holds for the complete mesh. The dual mesh $\tilde{\mathscr{G}}$ is not a cell-complex, because at the boundary $\partial\Omega$ boundary cells are missing. To be able to derive conservation statements up to the boundary we need to complete the dual mesh to a cell-complex. This can be done in the following way. Let us denote the mesh elements of the primal mesh that make up the boundary $\partial\Omega$ by $\mathscr{G}_b$. We take the dual mesh to $\mathscr{G}_b$ within $\partial\Omega$, which we denote by $\tilde{\mathscr{G}_b}$. The $(n-1)$-dimensional dual mesh $\tilde{\mathscr{G}_b}$ is a cell-complex because $\partial\partial\Omega = \emptyset$, and, moreover, the

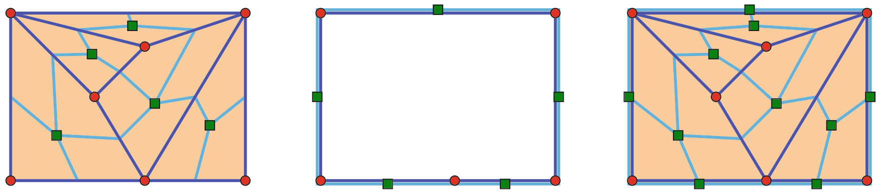

Fig. 3 On the *left*, we show a primal mesh $\mathscr{G}$ (*blue*/*red*) and its barycentric dual $\tilde{\mathscr{G}}$ (*light blue*/*green*). In the *middle*, we show the boundary part $\mathscr{G}_b$ of the primal mesh and the corresponding barycentric dual $\tilde{\mathscr{G}}_b$ within $\partial\Omega$. On the *right*, we show how $\tilde{\mathscr{G}}$ and $\tilde{\mathscr{G}}_b$ combine to form the cell-complex $\bar{\tilde{\mathscr{G}}}$

set $\tilde{\mathscr{G}}_b$ is exactly the set that completes $\tilde{\mathscr{G}}$ to a cell-complex $\bar{\tilde{\mathscr{G}}} := \tilde{\mathscr{G}} \cup \tilde{\mathscr{G}}_b$. This is illustrated in Fig. 3. We will subsequently partly define the variables and discretization on $\bar{\tilde{\mathscr{G}}}$, the fact that this is a cell-complex allows us to derive conservation statements for momentum and energy neatly up to the boundary.

The interpolation matrices between $C^{(k)}$ and $C^{\widetilde{(3-k)}}$ are called discrete Hodge operators, because they map discrete k-forms to discrete $(3-k)$-forms, like the continuous Hodge operators map differential k-forms to differential $(3-k)$-forms. When the circumcentric dual mesh is used, the orthogonality of the primal and dual mesh allow for consistent diagonal discrete Hodge operators [8]. When the barycentric dual mesh is used, diagonal discrete Hodge operators are still available for interpolation between $C^{(0)}$ and $C^{(\tilde{3})}$, and, $C^{(3)}$ and $C^{(\tilde{0})}$, by using the volume averages. However, for interpolation between primal lines and dual surfaces, and, primal surfaces and dual lines, no consistent diagonal Hodge operators exist for general polyhedra in the barycentric case.

As barycentric Hodge operators we will use the mimetic inner product matrices used in the mimetic finite difference method. It can be shown that the mimetic inner product matrices $M_{\mathscr{L}}$ and $M_{\mathscr{S}}$ presented in [2], which define the mimetic inner product on the spaces $C^{(1)}$ and $C^{(2)}$, respectively, can be interpreted as discrete Hodge operators mapping from $C^{(k)}$ to $C^{(3-k)}$, for $k = 1, 2$ for a barycentric dual mesh. It is shown in [14] that these matrices are consistent and stable for polyhedral meshes with only minor assumptions. We use the notation $\mathbb{H}^{(\tilde{2},1)}$ and $\mathbb{H}^{(\tilde{1},2)}$ for, respectively, $M_{\mathscr{L}}$ and $M_{\mathscr{S}}$ to indicate that they map from the primal mesh to the dual mesh. In the parts of the mesh where the primal and dual mesh are orthogonal we will use the diagonal Hodge operator instead.

These Hodge operators are symmetric and positive definite for mesh cells with varying number of faces and therefore allow for a generalization of the method described in [11] from simplicial meshes to polyhedral meshes.

3 Barycentric Discretization

We use a barycentric dual mesh and a discretization similar to [11], but we discretize the viscous term differently by employing the aforementioned Hodge operators. Moreover, we define the pressure at all the points in the dual cell-complex $\tilde{\bar{\mathscr{G}}}$ and solve an extra equation for the faces in $\mathscr{G}_b$, which allows conservation statements that hold up to the boundary. Furthermore we introduce extra vorticity variables only for the primal edges in the regions where the Hodge operator $\mathbb{H}^{(\tilde{2},1)}$ is not diagonal. This results in an efficient treatment of the nonorthogonal part of the mesh by avoiding the inversion of $\mathbb{H}^{(\tilde{2},1)}$ while still allowing for nonorthogonal polyhedral meshes. We discretize (1a) by approximating its line integral over dual line elements.

To define the convection operator we need a primal volume vector reconstruction operator that approximates the velocity vector in the barycenter of the primal volumes v by using the flux variables $u_s^{(2)}$ for $s \in \partial v$. The integral of the velocity field over the primal cells, can be approximated [12] according to

$$\int_v \underline{u}\, \mathrm{d}V \cong \sum_{s\in\partial v} \alpha_v(s)(\underline{x}_s - \underline{x}_v)u_s^{(2)},$$

where $\underline{x}_s$ and $\underline{x}_v$ are barycenters of s and v, respectively. This is a first order approximation that holds for arbitrary polygons and is a second order approximation when the mesh is uniform [12]. It can be written as $\mathbb{R}\mathbf{u}^{(2)}$, where $\mathbb{R}$ is the $3N_v \times N_s$ matrix with $\mathbb{R}^i_{v,s} = \alpha_v(s)(x^i_s - x^i_v)$, where $i = 1, 2, 3$ indicates the vector components. Let $\underline{\mathbb{H}}^{(\tilde{0},3)}$ be the $3N_v \times 3N_v$ matrix that divides the vector components by the cell volume of the corresponding cell. Then $\underline{\mathbb{H}}^{(\tilde{0},3)}\mathbb{R}\mathbf{u}^{(2)}$ gives an, in general first order, approximation of the vector field $\underline{u}$ in the dual points $\tilde{v} \in \tilde{\mathscr{V}}$, denoted by $\underline{u}_{\tilde{v}}^{(\tilde{0})}$. Given a vector field $\underline{c}$ discretized in dual mesh points $\tilde{v} \in \tilde{V}$ as $\underline{c}^{(\tilde{0})}$, an approximation of the line integral of this vector field over the dual lines $\tilde{s} \in \tilde{\mathscr{S}}$ is then given by [12]

$$\int_{\tilde{s}} \underline{c} \cdot \mathrm{d}\underline{l} \cong \sum_{\tilde{v}\in\partial\tilde{s}} \alpha_{\tilde{s}}(\tilde{v})\underline{c}_{\tilde{v}}^{(\tilde{0})} \cdot (\underline{x}_s - \underline{x}_v). \tag{2}$$

Outer orientations of $\tilde{s}$ and $\tilde{v}$ agree if and only if the orientations of s and v agree, hence (2) is written for all dual mesh lines at once as $\mathbb{R}^T\underline{\mathbf{c}}^{(\tilde{0})}$. These approximation properties of $\mathbb{R}$ and $\mathbb{R}^T$ will be used in the discretization of the convection term.

Integrating the convective term over primal cells and applying the divergence theorem we obtain

$$\int_v (\underline{u} \cdot \nabla)\underline{u}\, \mathrm{d}V = \sum_{s\in\partial v} \alpha_v(s) \int_s \underline{u}\, \underline{u} \cdot \mathrm{d}\underline{A}. \tag{3}$$

The surface integrals will be approximated as

$$\int_s \underline{u}\,\underline{u}\cdot \mathrm{d}\underline{A} \cong \sum_{\{v|s\in\partial v\}} \frac{1}{2}\underline{u}_{\tilde{v}}^{(\tilde{0})} u_s^{(2)}.$$

Let $\underline{\mathbf{u}}^{(\tilde{0})} = \underline{\mathbb{H}}^{(\tilde{0},3)}\mathbb{R}\mathbf{u}^{(2)}$ and let $\mathbb{A}[\underline{\mathbf{u}}^{(\tilde{0})}]$ be the $3N_s \times N_s$ matrix consisting of the three $N_s \times N_s$ diagonal blocks with on the diagonal element corresponding to s, respectively the x-, y- or z-component of $\sum_{\{v|s\in\partial v\}} \underline{u}_{\tilde{v}}^{(\tilde{0})}/2$. Using this matrix we can write the approximation of (3) for all $v \in \mathscr{V}$ at once as $\underline{\mathbb{D}}^{(3,2)}\mathbb{A}[\underline{\mathbf{u}}^{(\tilde{0})}]\mathbf{u}^{(2)}$, where $\underline{\mathbb{D}}^{(3,2)}$ is a componentwise version of $\mathbb{D}^{(3,2)}$. Subsequently applying $\underline{\mathbb{H}}^{(\tilde{0},3)}$ and $\mathbb{R}^T$ gives $\mathbb{C}[\mathbf{u}^{(2)}]\mathbf{u}^{(2)} := \mathbb{R}^T\underline{\mathbb{H}}^{(\tilde{0},3)}\underline{\mathbb{D}}^{(3,2)}\mathbb{A}[\underline{\mathbf{u}}^{(\tilde{0})}]\mathbf{u}^{(2)}$, which is an approximation of the convection term integrated over the dual line integrals.

For the primal lines $l \in \mathscr{L}$ for which $\mathbb{H}^{(\tilde{2},1)}$ contains off-diagonal terms we introduce the vorticity variables $\omega_l^{(1)}$. We collect them in a vector $\boldsymbol{\omega}^{(b1)}$, where b in the superindex indicates that the complete barycentric Hodge operator applies. To split the primal lines in the set where $\mathbb{H}^{(\tilde{2},1)}$ is diagonal and the set where it is non-diagonal we introduce matrices $\mathbb{I}_b : C^{(1)} \to C^{(b1)}$ and $\mathbb{I}_d : C^{(1)} \to C^{(d1)}$, where $\mathbb{I}_b$ is the identity with the rows corresponding to the lines where $\mathbb{H}^{(\tilde{2},1)}$ is the diagonal eliminated and $\mathbb{I}_d$ the same but then with the rows corresponding to lines where $\mathbb{H}^{(\tilde{2},1)}$ is non-diagonal eliminated.

For the pressure term we use a straightforward discretization by applying the discrete gradient operator $\bar{\mathbb{D}}^{(\tilde{1},\tilde{0})}$, where the bar indicates that this is the extension of $\mathbb{D}^{(\tilde{1},\tilde{0})}$ to the dual cell-complex. The complete semi-discrete system is given by

$$\begin{bmatrix} \mathbb{H}^{(\tilde{1},2)}\partial_t + \mathbb{C}[\mathbf{u}^{(2)}] + \nu\mathbb{L} & \mathbb{H}^{(\tilde{1},2)}\mathbb{D}^{(2,1)}\mathbb{I}_b^T & -\bar{\mathbb{D}}^{(\tilde{1},\tilde{0})} \\ \mathbb{I}_b\mathbb{D}^{(\tilde{2},\tilde{1})}\mathbb{H}^{(\tilde{1},2)} & \nu^{-1}\mathbb{I}_b\mathbb{H}^{(\tilde{2},1)}\mathbb{I}_b^T & 0 \\ -\bar{\mathbb{D}}^{(\tilde{1},\tilde{0}),T} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(2)} \\ \boldsymbol{\omega}^{(b1)} \\ \mathbf{p}^{(\tilde{0})} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^{(\tilde{1})} \\ \mathbf{r}_2^{(b1)} \\ \mathbf{r}_3^{(3)} \end{bmatrix},$$

where $\mathbb{L} := \mathbb{H}^{(\tilde{1},2)}\mathbb{D}^{(2,1)}\mathbb{I}_d^T(\mathbb{I}_d\mathbb{H}^{(\tilde{2},1)}\mathbb{I}_d^T)^{-1}\mathbb{I}_d\mathbb{D}^{(\tilde{2},\tilde{1})}\mathbb{H}^{(\tilde{1},2)}$ and it should be noted that $\mathbb{I}_d\mathbb{H}^{(\tilde{2},1)}\mathbb{I}_d^T$ is diagonal. The right-hand side vector incorporates the Dirichlet boundary condition on the velocity. In this semi-discrete system the viscous term in the momentum equation consists of two contributions $\nu\mathbb{L}\mathbf{u}^{(2)}$ and $\mathbb{H}^{(\tilde{1},2)}\mathbb{D}^{(2,1)}\mathbb{I}_b\boldsymbol{\omega}^{(b1)}$ corresponding to the orthogonal and non-orthogonal parts of the mesh, respectively.

From these discrete equations we can derive the discrete conservation laws that correspond to

$$\partial_t \int_\Omega \underline{u}\,\mathrm{d}V = -\int_{\partial\Omega} \underline{u}(\underline{u}\cdot\underline{n}) + p\underline{n} - \nu\nabla\underline{u}\cdot\underline{n}\,\mathrm{d}A,$$

$$\partial_t \int_\Omega \frac{1}{2}(\underline{u}\cdot\underline{u})\,\mathrm{d}V = -\int_{\partial\Omega} \frac{1}{2}\underline{u}\cdot\underline{u}(\underline{u}\cdot\underline{n}) + p(\underline{u}\cdot\underline{n}) + \nu(\underline{\omega}\times\underline{u})\cdot\underline{n}\,\mathrm{d}A - \nu\int_\Omega \underline{\omega}\cdot\underline{\omega}\,\mathrm{d}V.$$

Details will be given at FVCA8 and in a forthcoming publication.

4 Future Work

We have presented a mimetic staggered discretization of the incompressible Navier–Stokes equations. The method uses the barycentric dual mesh and discrete exterior calculus. This discretization will be used to develop a cut-cell method for modeling flow around complex objects by using a Cartesian mesh. The resulting efficient method, despite using a mesh not aligned with the objects, is anticipated to still be physically accurate as a result of its many conservation properties.

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